

Quantum Mechanics in Curved Configuration Space

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Different approaches are compared to formulation of quantum mechanics of a particle on the curved spaces. At first, the canonical, quasiclassical, and path integration formalisms are considered for quantization of geodesic motion on the Riemannian configuration spaces. A unique rule of ordering of operators in the canonical formalism and a unique definition of the path integral are established and, thus, a part of ambiguities in the quantum counterpart of geodesic motion is removed. A geometric interpretation is proposed for noninvariance of the quantum mechanics on coordinate transformations. An approach alternative to the quantization of geodesic motion is surveyed, which starts with the quantum theory of a neutral scalar field. Consequences of this alternative approach and the three formalisms of quantization are compared. In particular, the field theoretical approach generates a deformation of the canonical commutation relations between operators of coordinates and momenta of a particle. A cosmological consequence of the deformation is presented in short.

KEY WORDS: quantum mechanics; Riemannian space; geodesic motion; deformation.

1. INTRODUCTION

Quantum mechanics on the Riemannian geometric background is the simplest part of the fundamental problem of association of general relativity and the quantum theory. In the quantum mechanics, the problem of definition of appropriate physical observables appears in a relatively simple form, which emerges quite completely in quantization of gravitation (see, for instance, Rovelli, 1999). On the other hand, the quantum mechanics of a point-like particle may be considered as a limiting case of the string dynamics. It provides also a description of interesting physical models such as a motion on homogeneous spaces of some groups (see, for instance, Groshe *et al.*, 1997; Marinov, 1995). An important point is that the Quantum Field Theory in curved space-times which is applied successfully to describe fundamental processes in the early Universe is based in fact on some quantum mechanics of a (quasi-)particle, at least, implicitly (see Gibbons

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and Pohle, 1993; Tagirov, 1999). At last, one may expect that a modification of the well-established fundamental theory, such as quantum mechanics, to a more general geometrical background can reveal new features of the theory and serve for better understanding of it.

The problem has a long history related to the names of Podolsky, Dirac, DeWitt, and other less known theorists. Nevertheless, it has not still a satisfactory unambiguous solution. The main approach is based on the idea of quantization of the classical Hamiltonian systems and their generalizations. (In the simplest expressions, the quantization is a map of a classical theory in terms of the usual functions on a phase space to a mathematical structure in terms of noncommuting objects along definite aggregates of rules (*formalisms*), which depends on a small parameter \hbar the physical value of which is the Plank constant.)

There is a number of different formalisms of quantization and it is natural to expect that they give similar results for the same physical system. Unfortunately, it is not the case even for such an elementary system as the point-like chargeless and spinless particle if the configuration space is curved. Moreover, as a rule, there are fundamental ambiguities even in the framework of the same formalism and even for a simple class of phase spaces $\mathcal{P}_{2n} \sim R_n \otimes V_n$ where V_n is the n -dimensional Riemannian configuration space.

In the canonical and path integration formalisms, see Sections 2 and 4 respectively; the ambiguities appear in the following two forms. The first one is the known problem of ordering of operators $\hat{\xi}^i, \hat{p}_j, i, j, \dots = 1, \dots, n$, which correspond to the Darboux coordinates ξ^i, p_j in \mathcal{P}_{2n} when they are substituted into a function $f(\xi, p)$ (say, through a power expansion) to obtain the corresponding quantum observable \hat{f} or the path integral. Generally, there is no leading principle to single out a certain rule of ordering among infinitely many ones. The ambiguity does not attract much attention in view of that all of the rules lead to the same operator \hat{f} up to an additive constant if $f(\xi, p) = f_0(\xi, p) + f_1(\xi) + f_2(p)$, where f_0 is a second-order polynomial of the Darboux coordinates ξ, p and f_1, f_2 are appropriate arbitrary functions. The classical Hamiltonians of the typical problems of the standard quantum mechanics in the Euclidean configuration space E_n are in this class if the preferred Cartesian coordinates are taken as ξ^i . The latter is usually assumed with no stipulation. The curvilinear coordinates are used, if any, *a posteriori*, only as a technical tool, for example, in relation with a symmetry of the potential. However, even in E_n , as soon as curvilinear coordinates are taken as *observables*, i.e., as one half of the phase space coordinates, then f_0 that was a second-order polynomial in the Cartesian coordinates and their conjugate momenta, fails generally to be a polynomial at all. Respectively, the dependence of quantization on a choice of ordering becomes actual. In addition, for the path integration, there is an ambiguity in the choice of the points on a lattice of integration, in which the integrands are evaluated (see, for instance, D'Oliveo and Torres, 1989; and Section 4 below). It is a common problem for any geometry of

the configuration space and system of coordinates, again except the case of E_n , Cartesian coordinates and the quadratic function f_0 .

The second ambiguity consists in that the result of such quantization depends on the choice of coordinates ξ^i in V_n , that is not invariant with respect to diffeomorphisms of V_n , or, *diffeononinvariant* even if a rule of ordering is fixed, although the original classical theory is diffeoinvariant. Again, in the standard theory, the problem is obscured by existence of the Cartesian coordinates. It looks as there were an implicit postulate that quantization should be performed just in these preferred coordinates. However, what should one do in V_n , where the Cartesian coordinates do not exist at all? An attempt to answer on the question will be given below in Section 5 on the basis of results of the preceding sections.

According to Bordemann *et al.* (1998), the deformation quantization in the framework of the Fedosov formalism (see Fedosov, 1994), leads to diffeoinvariant quantum mechanics in V_n . However, this result is obtained by using a particular rule of ordering, namely, Weyl's one. Thus, at least, the ambiguity in ordering apparently retains.

Geometric quantization in the Blattner–Costant–Souriau formalism (see, for instance, Abraham and Marsden, 1978; Śniatycki, 1980), is reduced to the quasiclassical approach by Pauli–DeWitt (Pauli, 1950–51; DeWitt, 1957) for the simple case under our consideration. The formalism is diffeoinvariant and includes no ordering procedure, but it is approximate *ab initio* because it starts with an *Ansatz* where the (unknown) quantum propagator is substituted by the quasiclassical one.

Among other approaches to quantization on V_n , it is worth to mention the one based on embedding V_n to an Euclidean space of a greater dimension and using the Cartesian coordinates in it (Ogawa *et al.*, 1990). And, at last, the present author develops an approach to quantum mechanics of a particle in V_n , which is an alternative to quantization of mechanics and may be called the quantum-field-theoretical one, or the *QFT-approach* (Tagirov, 1990, 1992, 1996, 1999). It reproduces quantum mechanics in the general $V_{1,n}$ in a diffeoinvariant and ordering-independent form as the quasi-nonrelativistic asymptotic of a quasi-one-particle sector on an appropriate Fock space for the quantized neutral scalar field. (In the paper by Tagirov [1996], the field of spin 1/2 is considered but the result needs some refinement and justification along the lines of Tagirov [1999] and Section 6 of the present paper.) Thus, in this approach, the canonical quantization procedure is shifted from the particle phase space to the quantization of the field. The diffeoinvariant analogs of the operators $\hat{\xi}$, \hat{p} mentioned earlier prove to satisfy a deformation² of the canonical commutation relations such that the position operators mutually do not commute; of course, the conjugate momenta are also mutually noncommutative.

²The term “deformation” is used very deliberately in the present paper to denote a substitution of the Poisson or Lie bracket by an asymptotic sum the terms of which are bilinear and antisymmetric in the same sense as the brackets themselves are; this is only one of the properties of the notion of deformation used in the mathematically more rigorous texts.

The deformation parameter is c^{-2} . Just this and other curious results of the approach stimulated this author's interest to the state of art in the traditional approaches to quantum mechanics in V_n .

In this paper the three historically first formalisms of quantization, the canonical, quasiclassical ones, and the path integration, will be considered in application to the geodesic motion in configuration space V_n with the general time-independent metric tensor $\omega_{ij}(\xi)$. The latter means that the *space-time* is $V_{1,n} \sim R_1 \otimes V_n$. The Hamilton operators arising in the three formalisms are compared in a certain approximation in which they should come to the same Hamilton operator. This condition distinguishes a unique rule of ordering of the primary observables operators for the canonical and path integration formalisms and gives an unambiguous prescription for the latter. (Along the reasons mentioned earlier, these two formalisms are considered below as "more exact" ones with respect to the quasiclassical one.)

We postpone the deformation quantization approach and embedding of V_n for more serious special consideration though use the general idea on deformation of the Poisson brackets in a formulation of postulates of canonical operator formalism in Section 2.

In Section 2, it will be shown that, for the canonical quantization of the geodesic motion in V_n , the freedom in the choice of ordering rules is reduced to a one-parametric set in each fixed system of coordinates $\{\xi^i\}$. Since diffeomorphisms of V_n are determined by n arbitrary C^∞ -functions, one may say figuratively that the overall arbitrariness is " $1 + \infty^3$ -dimensional" in this case.

In Section 3, it is shown that the one-dimensional part of the arbitrariness can be removed by condition of coincidence of the canonical Hamilton operator with that by DeWitt (1957) in a certain approximation.

In Section 4, the path integral for the quantum propagator of geodesic motion is constructed so that the phase of the integrand is proportional to the classical action and the Hamilton operator generating the propagator coincides with DeWitt's one in the same approximation, as in the canonical case. This fixes the same rule of ordering of the primary operators as in Section 3 and unambiguously determines that the integrands should be evaluated at the nodes of the lattice of integration.

In Section 5, the obtained solution of the problem of ordering is discussed and a possible explanation of the diffeononinvariance of the canonical quantum mechanics in V_n is given.

In Section 6, a survey of main results of the above-mentioned QFT-approach is given and compared with the results of quantization of mechanics.

The paper adopts the so-called heuristic (or, naive) level of mathematical rigor: many definitions and relations need further refinement to have an exact meaning. It is expected that the latter can be achieved if physically sensible results appear at our imperfect level.

2. CANONICAL QUANTIZATION OF GEODESIC MOTION IN THE RIEMANNIAN CONFIGURATION SPACE

2.1. Hamilton Theory of Geodesic Motion

To emphasize a relation of the system under consideration to general relativity, let us start with geodesic lines in the generic $(1 + n)$ -dimensional Riemannian space-time $V_{1,n}$ of the Lorentz signature $-n + 1$. Let x^α , $(\alpha, \beta, \dots = 0, 1, \dots, n)$ be some coordinates in $V_{1,n}$, and t, ξ^i , $(i, j, \dots = 1, 2, \dots, n)$ be normal Gaussian coordinates generated by the normal geodesic translation of a given Cauchy hypersurface $\Sigma \equiv \Sigma(t_0)$ and some coordinates ξ^i on it. The metric form is

$$\begin{aligned} ds^2 &= g_{\alpha\beta} dx^\alpha dx^\beta \\ &= c^2 dt^2 - \omega_{ij}(t, \xi) d\xi^i d\xi^j \quad t \in [t_0, t_1]. \end{aligned} \tag{1}$$

(The range where the coordinates t, ξ^i and the representation of the metric (1) are valid is indicated, for instance, by Destri *et al.* (1994, Section 2).)

The space-time geodesic lines are extremals $x^\alpha = x^\alpha(s)$ of the action functional

$$W = -mc \int_{s_1}^{s_2} ds \sqrt{g_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds}} \stackrel{def}{=} \int_{s_1}^{s_2} L' ds = \int_{t_1}^{t_2} L dt, \tag{2}$$

which satisfy the following constraint:

$$g^{\alpha\beta}(x) p_\alpha p_\beta = m^2 c^2, \tag{3}$$

where p_α are the generalized momenta

$$p_\gamma(s) \stackrel{def}{=} \frac{dL'}{d(dx^\gamma/ds)} = -mc \left(g_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} \right)^{-1/2} g_{\gamma\delta} \frac{dx^\delta}{ds}. \tag{4}$$

The canonical quantization as a map Q of functions on a phase space $\mathcal{P} = \mathcal{P}_{2n+2} \sim T^*V_{1,n}$ to operators acting on a Hilbert space \mathcal{H} (see a more exact definition below) can be applied to this diffeoinvariant system with constraint (3). However, it would be a map on operators acting on the space³ $\mathcal{H} \sim L^2(V_{1,3}; \mathbb{C}; \sqrt{g}d_4x)$ which cannot be interpreted as a space of states of a real particle specified by a position in the configuration space (see Tagirov, 1999). For the standard probability interpretation in the Schrödinger representation, the operators of observables should be defined on $L^2(\Sigma; \mathbb{C}; \sqrt{\omega}d^n x)$. It is realized by quantization of the reduced Hamiltonian

³Here and further the spaces $L^2(V_{1,3})$ and $L^2(V_n)$ are defined with respect to the natural measures induced by the corresponding Riemannian metric forms. This allows to consider the functions from these spaces as scalars with respect to the diffeomorphisms in $V_{1,n}$ and V_n . If there were no metric, a more complicate construction with a class of the equivalent Lebesgue measures on the configuration space and the half forms instead of the scalars should be used (see Abraham and Marsden, 1978, p. 427).

system in which one solves the constraint (3) at the classical level. To this end, one represents (3) in the form

$$\left(p_0 + mc\sqrt{1 + \frac{2H_0}{mc^2}}\right)\left(p_0 - mc\sqrt{1 + \frac{2H_0}{mc^2}}\right) = 0 \quad (x^0 \equiv ct), \quad (5)$$

where

$$H_0 \equiv H_0(\xi, p; t) \stackrel{\text{def}}{=} \frac{1}{2m}\omega^{ij}(\xi; t)p_i p_j. \quad (6)$$

For the nonradiating and spinless particle (just only it moves along a geodesic line in the Riemannian space-time $V_{1,n}$), we shall take, as usually, the solution of 5 with respect to p_0 such that $p_0 > 0$. Then, in the theory with the constraint thus resolved, the Hamilton function will be

$$H(\xi, p) = mc^2\sqrt{1 + \frac{2H_0}{mc^2}}. \quad (7)$$

There is an interesting intermediate approach of Gitman and Tyutin (1990) in which both the solutions of constraint (5) are used (in $E_{1,3}$) through introduction of a special observable “the sign of p_0 .” This leads to a state space consisting of two $L^2(E_{1,3}; \mathbb{C}; d_3x)$, $\{x^i\} \in E_{1,3}$, which describe particles and antiparticles respectively (being neutral, they are identical). Gavrilov and Gitman (2001) have extended the approach to the case of $V_{1,3}$. However, a remark arises concerning this work, which will be made at the end of the present section, near the formula (29).

The nonreduced and reduced formalisms differ in that, in the former case, a time-like coordinate x^0 is included to the set of observables whereas, in the latter case, the variable t is an evolution parameter. In the classical theory, these formalisms are physically equivalent versions of the same theory; however, quantization of them leads to different theories.

In the reduced formalism, observables for fixed t are functions on the phase space $\mathcal{P}_{2n} \sim T^*\Sigma(t)$, the cotangent bundle over $\Sigma(t)$. They may be considered locally as functions of Darboux coordinates ξ^i, p_j , with $\{\xi^i\} \in \Sigma(t)$ and

$$p_k = \frac{mc\omega_{kl}(t, \xi)\xi^l}{\sqrt{c^2 - \omega_{ij}(t, \xi)\xi^i\xi^j}}. \quad (8)$$

Of course, the Darboux coordinates fixed by a choice of coordinates ξ^i on $\Sigma(t)$ are observables, too. They form the so-called *primary observables* in the sense that other observables are functions of them.

2.2. General Concept of Canonical Quantization

Consider now the concept of quantization of a classical Hamiltonian system. A general definition of the asymptotical quantization can be found in the book by Karasev and Maslov (1991, chap. IV). We shall adopt the following simplification of the deformational version of this definition. (The simplification consists in that the deformed Poisson brackets are supposed in condition (Q2) below instead of the usual definition which starts with a $*$ -product of symbols of operators.)

Let s_{2n} be an appropriate subalgebra of the Poisson algebra of functions $f \in C^\infty(\mathcal{P}_{2n})$.

Quantization is a map

$$\mathcal{Q} : s_{2n} \ni f \xrightarrow{\mathcal{Q}} \hat{f} \text{ (operators in a Hilbert space } \mathcal{H}\text{)}, \tag{9}$$

satisfying the following conditions:

- (Q1) $1 \xrightarrow{\mathcal{Q}} \hat{1}$ (the identity operator in \mathcal{H});
- (Q2) $\{f, g\}_\hbar \xrightarrow{\mathcal{Q}} i\hbar^{-1}[\hat{f}, \hat{g}] \stackrel{def}{=} i\hbar^{-1}(\hat{f}\hat{g} - \hat{g}\hat{f})$ where $\{f, g\}_\hbar \equiv \{f, g\}_0 + O(\hbar)$ is an antisymmetric bilinear functional of f and g and $\{f, g\}_0 \equiv \{f, g\}$ is the Poisson bracket in \mathcal{P}_{2n} ;
- (Q3) $\hat{f} \xrightarrow{\mathcal{Q}} (\hat{f})^\dagger$ (the Hermitean conjugation of \hat{f} with respect to the scalar product in \mathcal{H});
- (Q4) a complete set of functions (maximal Abelian subalgebra) $f^{(1)}, \dots, f^{(n)}$: $f^{(i)} \in s_{2n}$, is mapped to a complete set (in the sense by Dirac (1948, chap. III) of commuting operators $\hat{f}^{(1)}, \dots, \hat{f}^{(n)}$.

It follows also from the condition (Q4) and the Stone-Von Neumann theorem that $\mathcal{H} \sim L^2(\Sigma; \mathbb{C}; \sqrt{\omega}d^n x)$.

The main problems of quantization consist in an infinite number of possibilities to construct the functional $\{f, g\}_\hbar$ (deformation of the Poisson bracket), in difficulties with construction of a complete set on the topologically nontrivial spaces \mathcal{P}_{2n} and in diffeononinvariance of quantum observables. Here we have a simple and physically oriented purpose to consider traditional procedures of quantization in application to a particular elementary system on a class of simple but nontrivial geometric backgrounds. Therefore, the following restrictions on the system and spaces $V_{1,n}$ and V_n under consideration will be supposed.

- (V1) Assume that $V_{1,n}$ is a globally static space-time and $\Sigma(t) \sim V_n$ are its completely geodesic sections that exist in this case. It means that $\omega_{ij}(\xi, t) \equiv \omega_{ij}(\xi)$. Then, the classical dynamics with the Hamilton functions H and H_0 are equivalent and refer only to different systems of reference.
- (V2) Our main purpose is to construct a quantum image of the Hamilton function (classical Hamiltonian) H_0 for an arbitrary $\omega_{ij}(\xi) \in C^\infty(V_n)$ starting with

the general scheme of quantization (Q1)–(Q2). The minimal algebra s_{2n} containing all such Hamiltonians is the algebra of polynomials in p_i with the coefficients depending on ξ^i . If a nonrelativistic quantum Hamiltonian \hat{H}_0 is constructed, then a possible way to obtain a relativistic one \hat{H} is provided by the Von Neumann rule (Von Neumann, 1955, p. 313) defining functions of commuting operators $\hat{A}_1, \dots, \hat{A}_N$:

$$f(A_1, \dots, A_N) \xrightarrow{\mathcal{Q}} \hat{f} \stackrel{\text{def}}{=} f(\hat{A}_1, \dots, \hat{A}_N). \quad (10)$$

Being applied to the classical Hamiltonian (7) interpreted in the asymptotical sense, it gives

$$\hat{H}(H_0) = H(\hat{H}_0) \stackrel{\text{def}}{=} \hat{H}_0 - \frac{1}{mc^2} \hat{H}_0^2 + \frac{1}{2m^2c^4} \hat{H}_0^4 - \dots \quad (11)$$

- (V3) Assume that the topology of V_n is trivial; of course, it does not mean that the curvature of V_n along the metric ω_{ij} is zero. The physical meaning of this condition may not be considered as a restriction on the topology but as localization of the quantum particle in a sufficiently small domain so that only local manifestations of the space curvature are essential.
- (V4) In virtue of the preceding assumption, it is supposed that the coordinate lines ξ^i on V_n are complete and open. In this sense, they are similar to the Cartesian coordinates.

By the way, under assumptions (V1)–(V4), there are no QFT process of creation and annihilation of particles by the external gravitational field, and the quantum dynamics becomes a purely quantum–mechanical one.

The canonical quantization means here the following realization of \mathcal{Q} :

- (CQ1) One takes some coordinates ξ^i satisfying (V4) as a complete set $f^{(1)}, \dots, f^{(n)}$ in the condition (Q4).
- (CQ2) One takes, at first, the algebra of polynomials in the Darboux coordinates ξ^i, p_j as the algebra s_{2n} .
- (CQ3) One imposes the following conditions on the functional $\{f, g\}_{\hbar}$:

$$\{\xi^i, \xi^j\}_{\hbar} = \{\xi^i, \xi^j\}_0 \equiv \{\xi^i, \xi^j\} = 0, \quad (12)$$

$$\{\xi^i, p_j\}_{\hbar} = \{\xi^i, p_j\}_0 \equiv \{\xi^i, p_j\} = \delta_j^i, \quad (13)$$

$$\{p_i, p_j\}_{\hbar} = \{p_i, p_j\}_0 \equiv \{p_i, p_j\} = 0, \quad (14)$$

thus making the condition (Q2) more definite.

(CQ4) Then, the quantum images $\hat{\xi}^i, \hat{p}_j$ of these primary classical observables should satisfy the canonical commutation relations

$$[\hat{\xi}^i, \hat{\xi}^j] = 0, \quad [\hat{\xi}^i, \hat{p}_j] = i\hbar\delta_j^i, \quad [\hat{p}_i, \hat{p}_j] = 0. \quad (15)$$

and may be realized as differential operators⁴

$$\hat{\xi}^i \stackrel{Q}{\rightarrow} \hat{\xi}^i = \xi^i \cdot \hat{\mathbf{1}}, \quad \hat{p}_j \stackrel{Q}{\rightarrow} \hat{p}_j = -i\hbar \left(\partial_j + \frac{1}{4} \partial_j \ln \omega \right). \quad (16)$$

in $L^2(\Sigma(= V_n); \mathbb{C}; \sqrt{\omega} d^n \xi)$.

(CQ5) Further, one maps the basis of s_{2n} formed by the unity and monomials

$$(\hat{\xi}^1)^{M_1} \dots (\hat{\xi}^n)^{M_n} (p_1)^{N_1} \dots (p_n)^{N_n} \quad (17)$$

onto the identity operator $\hat{\mathbf{1}}$ and hermitizations of the same monomials formed by operators $\hat{\xi}^i, \hat{p}_j$ along a chosen *rule of ordering* (a Hermitean arrangement of the operators in each monomial). As usual, the rule determines the functional $\{f, g\}_\hbar$ in the quantization condition (Q2) via commutation of the operators thus obtained, although it is not known to the present author if any rule determines the functional.

(CQ6) The functional $\{f, g\}_\hbar$ fixed by a rule of ordering is taken further as the general relation (Q2) for any $f(\xi, p), g(\xi, p) \in C^\infty(\mathcal{P}_{2n})$ in view of the density of the polynomials in C^∞ (see Berezin and Shubin, 1984).

In general, there are infinitely many possible rules of ordering and a classification of them, apparently not exhausting, is given by Agarwal and Wolf (1970).

The *Weyl rule* (Weyl, 1931) is the most popular one in the literature. It has some attractive symmetry properties (see, for instance, Mehta, 1964). For the particular case under consideration, it may be described as follows. Consider, for example, the following product $(\hat{p}_1)^a (\hat{\xi}^1)^b, a \geq 0, b \geq 0$ of noncommuting operators. Then, the Weyl ordering $((\hat{p}_1)^a (\hat{\xi}^1)^b)^{(W)}$ of the product is determined by the following relation (Berezin and Shubin, 1984, chap. 5):

$$(A\hat{p}_1 + B\hat{\xi}^1)^N = \sum_{a+b=N} \frac{N!}{a!b!} A^a B^b ((\hat{p}_1)^a (\hat{\xi}^1)^b)^{(W)}. \quad (18)$$

If one takes the Weyl ordering, then the functional $\{f, g\}_\hbar \stackrel{def}{=} \{f, g\}_\hbar^{(W)}$ is the Moyal bracket (Moyal, 1949). Taking into account the Riemannian measure on Σ and condition (V3) on coordinates ξ , one can represent the canonical quantization of

⁴It is well-known that the operators $\hat{\xi}^i, \hat{p}_j$ are symmetrical (we shall call *Hermitean*) but not self-adjoint ones in $L^2(V_n; \mathbb{C}; \sqrt{\omega} d^n \xi)$. A consecutive solution of this problem is achieved by introducing the rigged Hilbert space (see, for instance, Sudbery, 1986). Here, we shall adopt a more simple assumption that only an appropriate dense subset in $L^2(V_n; \mathbb{C}; \sqrt{\omega} d^n \xi)$ is under consideration.

the polynomials $f(\xi, p)$ via the Weyl ordering as follows (Berezin and Shubin, (1984, chap. 5):

$$\begin{aligned}
 f(p, \xi) \overset{Q}{\rightarrow} (\hat{f}^{(W)}\psi)(\xi) &= (2\pi\hbar)^{-n} \omega^{-\frac{1}{4}}(\xi) \\
 &\times \int d^n \xi' d^n p \exp\left(-\frac{i}{\hbar}(\xi^i - \xi'^i)p_i\right) f\left(p, \frac{\xi + \xi'}{2}\right) \\
 &\times \omega^{\frac{1}{4}}(\xi')\psi(\xi'), \quad \psi(\xi) \in L^2(\Sigma; \mathbb{C}; \sqrt{\omega}d^n\xi). \tag{19}
 \end{aligned}$$

Further, in view of the mentioned density of the polynomials in $C^\infty(\mathcal{P}_{2n} \equiv R_n \otimes V_n)$, this correspondence is adopted as a general definition of the canonical quantization of $f(p, \xi) \in C^\infty(\mathcal{P}_{2n})$.

Another example is the Rivier rule of ordering (Mehta, 1964; Rivier, 1957) which, in application to a monomial (17), is the following arrangement of the primary observables:

$$\begin{aligned}
 (\xi^1)^{M_1} \dots (\xi^n)^{M_n} (p_1)^{N_1} \dots (p_n)^{N_n} &\overset{Q}{\rightarrow} \frac{1}{2}((\hat{\xi}^1)^{M_1} \dots (\hat{\xi}^n)^{M_n} (\hat{p}_1)^{N_1} \dots (\hat{p}_n)^{N_n} \\
 + (\hat{p}_1)^{N_1} \dots (\hat{p}_n)^{N_n} (\hat{\xi}^1)^{M_1} \dots (\hat{\xi}^n)^{M_n}) &\stackrel{def}{=} ((\hat{\xi}^1)^{M_1} \dots (\hat{\xi}^n)^{M_n} (\hat{p}_1)^{N_1} \dots (\hat{p}_n)^{N_n})^{(R)}. \tag{20}
 \end{aligned}$$

Similarly to the Weyl ordering, it can be represented in the form,

$$\begin{aligned}
 f(\xi, p) \overset{Q}{\rightarrow} (\hat{f}^{(R)}\psi)(\xi) &= (2\pi\hbar)^{-n} \omega^{-\frac{1}{4}}(\xi) \\
 &\times \int d^n \xi' d^n p \exp\left(-\frac{i}{\hbar}(\xi^i - \xi'^i)p_i\right) \frac{f(\xi, p) + f(\xi', p)}{2} \\
 &\times \omega^{\frac{1}{4}}(\xi')\psi(\xi'), \quad \psi(\xi) \in L^2(V_n; \mathbb{C}; \sqrt{\omega}d^n\xi), \tag{21}
 \end{aligned}$$

which is obtained as the half-sum of the integral representations of qp - and pq -orderings given by Berezin and Shubin (1984, chap. 5). Again, the rule (21) is extended to all $f(\xi, p) \in C^\infty(\mathcal{P}_{2n})$. To the Rivier ordering, its own “bracket” corresponds in the condition (Q2), which may be denoted as $\{f, g\}_\hbar^{(R)}$.

Rewrite (19) and (21) in a compact form

$$\begin{aligned}
 (\hat{f}^{(W)}\psi)(\xi) &= \int d^n \xi' K_f^{(W)}(\xi; \xi')\psi(\xi') \\
 \text{and } (\hat{f}^{(R)}\psi)(\xi) &= \int d^n \xi' K_f^{(R)}(\xi; \xi')\psi(\xi'). \tag{22}
 \end{aligned}$$

It is obvious that the kernels of the form

$$K_f^{(v)}(\xi, \xi') \stackrel{def}{=} v K_f^{(W)}(\xi, \xi') + (1 - v) K_f^{(R)}(\xi, \xi') \tag{23}$$

define an ordering for any fixed value of the real parameter ν , too, and, in general, there are many other possibilities of such linear combinations.

2.3. Quantization of Geodesic Motion

It is the time now to return to the concrete system we intend to quantize, namely, the system described by the Hamilton function $H_0(\xi, p)$. An important point is that we apply the Von Neumann rule (10) to the metric tensor $\omega_{ij}(\xi)$:

$$\omega^{ij}(\xi) \xrightarrow{Q} \hat{\omega} \equiv \omega^{ij}(\hat{\xi}) = \omega^{ij}(\xi) \cdot \hat{\mathbf{1}}. \tag{24}$$

Suppose also that *operators* of the form

$$(\partial_{i_1} \dots \partial_{i_M} \widehat{\omega}^{ij}(\xi)), \quad M > 0,$$

should not appear in the canonical Hamilton operator \hat{H}_0 , which we are looking for. It does not mean that the representation of \hat{H}_0 as a differential operator in $L^2(V_n; \mathbb{C}; \sqrt{\omega} d^n \xi)$ should not contain *functions* of the form $\partial_{i_1} \dots \partial_{i_M} \omega_{ij}(\xi)$. Then, it is easy to see that all possibilities to choose rules of ordering for quantization along the scheme (CQ1)–(CQ2) are reduced to a one-parametric family (23). Simply speaking, the possible orderings are all those Hermitean arrangements of the operators $\hat{p}_i \hat{\omega}^{jk}$, which reproduce the classical Hamiltonian H_0 under assumption that the operators commute and thus satisfy to the Correspondence Principle. However, if, for example, a system with a classical Hamiltonian of the form $\lambda^{ijkl}(\xi) p_i p_j p_k p_l$ were considered the one-parametric family of kernels (23) would not exhaust all possible orderings. The latter would form apparently a two-parametric family and, thus, the ambiguity became larger.

Quantization of H_0 along the rule (23) gives the following correspondence after use of (19) and (21):

$$H_0(q, p) \xrightarrow{Q} \hat{H}_0^{(\nu)} = \frac{2-\nu}{8m} \omega^{ij}(\hat{\xi}) \hat{p}_i \hat{p}_j + \frac{\nu}{4m} \hat{p}_i \omega^{ij}(\hat{\xi}) \hat{p}_j + \frac{2-\nu}{8m} \hat{p}_i \hat{p}_j \omega^{ij}(\hat{\xi}). \tag{25}$$

Substituting here representations (16) of the primary operators, one obtains $\hat{H}_0^{(\nu)}$ as a differential operator in $L^2(V_n; \mathbb{C}; \sqrt{\omega} d^n \xi)$

$$\hat{H}_0^{(\nu)} = -\frac{\hbar^2}{2m} \Delta_{(\omega)}(\xi) + V_q^{(\nu)}(\xi), \tag{26}$$

where $\Delta_{(\omega)}$ is the Laplace–Beltrami operator for V_n ,

$$V_q^{(\nu)}(\xi) = -\frac{\hbar^2}{4m} \left(\partial_i (\omega^{ij} \gamma_j) + \frac{\nu}{2} \partial_i \partial_j \omega^{ij} + \frac{1-\nu}{2} \omega^{ij} \gamma_i \gamma_j \right) \tag{27}$$

is the so-called *quantum potential* and

$$\gamma_i \stackrel{def}{=} \gamma_{ij}^j, \quad \gamma_{ij}^k \stackrel{def}{=} \frac{1}{2} \omega^{kl} (\partial_i \omega_{jl} + \partial_j \omega_{il} - \partial_l \omega_{ij}) \quad (28)$$

are the Christoffel symbols. Contrary to the kinetic term $\hat{H}_0^{(kin)} \stackrel{def}{=} -(\hbar^2/2m) \Delta_{(\omega)}(\xi)$, the quantum potential is not diffeoinvariant. In the generic case, there is no choice of coordinates ξ for which $V_q^{(\nu)}(\xi) \equiv 0$ in a domain; it is easily seen from consideration of the integrability condition of the equation $V_q^{(\nu)} = 0$. In this sense, the quantum potential distinguishes no preferred coordinate system. This dependence of the quantum dynamics on coordinate systems can be called *apparently a quantum anomaly of diffeomorphisms of the configuration space*.

Thus, the arbitrariness in construction of quantum mechanics of a particle in $V_{1,3}$ is contained in the quantum potential $V_q^{(\nu)}$ in the form of its dependence on the parameter ν and on a choice of coordinates ξ^i . This arbitrariness is not trivial because it leads to Hamilton operators with different spectra. Some authors eliminate it “by hand” setting simply $\hat{H}_0 \equiv \hat{H}_0^{(kin)}$. Just so Gavrilov and Gitman (2001) do in fact. They consider the space $L^2(V_n; \mathbb{C}; d^n \xi)$ and take there as \hat{H}_0 (in their own notation) the operator

$$\hat{H}_0^{(GG)} = \frac{1}{2m} \hat{\omega}^{-\frac{1}{4}} \hat{p}_i \hat{\omega}^{\frac{1}{2}} \hat{\omega}^{ij} \hat{p}_j \hat{\omega}^{\frac{1}{4}} \equiv -\frac{\hbar^2}{2m} \sqrt{\omega} \Delta_{(\omega)}, \quad (29)$$

which is equivalent to $H_0^{(kin)}$; here, of course, $\hat{\omega}^{ij} \equiv \omega^{ij}(\xi) \cdot \hat{\mathbf{1}}$. The correspondence principle is evidently satisfied: if one assumes that $\hat{\xi}$ and \hat{p} commute then he comes to H_0 . A problem, however, is to go a way in the reverse direction and to obtain the rule (29) as a Hamilton operator along a more or less well formulated quantization formalism. Representation (29) can be found in the paper by DeWitt (1957) but namely as the kinetic part of the total Hamilton operator which includes also a quantum potential. A brief exposition of this result and its application for elimination of the ambiguity of the canonical quantization described by the parameter ν will be given in the following chapter.

3. QUASICLASSICAL QUANTIZATION OF GEODESIC MOTION

3.1. Dewitt's Hamiltonian and Riemannian Coordinates

DeWitt (1957) generalized to V_n the WKB-propagator proposed by Pauli (1950–51) for a particle in the electromagnetic field in $E_{1,3}$. As a result, the following nonrelativistic propagator was obtained:

$$\langle \xi, t | \xi_0, t_0 \rangle = \omega^{-1/4}(\xi) D^{1/2}(\xi, t | \xi_0, t_0) \omega^{-1/4}(\xi) \exp\left(-\frac{i}{\hbar} S(\xi, t | \xi_0, t_0)\right), \quad (30)$$

where D is the Van Vleck determinant (Van Vleck, 1928)

$$D(\xi, t|\xi_0, t_0) \stackrel{def}{=} \det\left(-\frac{\partial^2 S(\xi, t|\xi_0, t_0)}{\partial \xi^i \partial \xi_0^j}\right), \tag{31}$$

and

$$S(\xi, t|\xi_0, t_0) = \int_{t_0}^t \frac{1}{2} \omega_{ij}(\xi, t) \dot{\xi}^i \dot{\xi}^j, \quad \xi_0 \stackrel{def}{=} \xi(t_0) \tag{32}$$

is the classical action; its minimum is provided by the following equation of motion:

$$\ddot{\xi}^i + \gamma_{kl}^i(\xi; t) \dot{\xi}^k \dot{\xi}^l + \omega^{ik}(\xi; t) \frac{\partial \omega_{kl}}{\partial t}(\xi; t) \dot{\xi}^l = 0. \tag{33}$$

If $\partial \omega_{kl}(\xi; t)/\partial t = 0$, that is, if $V_{1,n}$ is the globally static space-time and $\Sigma(t) \sim \Sigma(t_0) \sim V_n$ is a completely geodesic hypersurface, then (33) is the geodesic equation in V_n . Restrict our consideration to this simple case, the more so that DeWitt does, in fact, the same.

Considering the limit $t \rightarrow t_0 (\xi \rightarrow \xi_0)$ along the geodesic line, connecting ξ and ξ_0 , DeWitt comes to the equation

$$i\hbar \frac{\partial}{\partial t} \langle \xi | \xi_0 \rangle + \frac{\hbar^2}{2m} \left(\Delta_{(\omega)}(\xi) - \frac{1}{6} R_{(\omega)}(\xi) \right) \langle \xi | \xi_0 \rangle = o(\xi - \xi_0) \langle \xi | \xi_0 \rangle, \tag{34}$$

where $R_{(\omega)}$ is the scalar curvature for the metric ω_{ij} ; the Riemann–Christoffel and Ricci tensors being defined as follows:

$$R_{(\omega)abcd}^a = \partial_d \gamma_{bc}^a - \partial_c \gamma_{bd}^a + \gamma_{de}^a \gamma_{bc}^e - \gamma_{ce}^a \gamma_{bd}^e, \quad R_{(\omega)ij} = R_{(\omega)ikj}^k. \tag{35}$$

(DeWitt’s definition of $R_{(\omega)ij}$ has an opposite sign.) It follows from (34) that the differential operators

$$\hat{H}_0^{(DW)}(\xi) = -\frac{\hbar^2}{2m} \left(\Delta_{\omega}(\xi) - \frac{1}{6} R_{(\omega)}(\xi) \right), \tag{36}$$

can be considered as the Hamilton operator on the subspace of the wave functions (initial data for the Schrödinger equation)

$$\langle \xi | \xi_0 \rangle \equiv \psi_{\xi_0}(\xi) \in L^2(V_n; \mathbb{C}; \sqrt{\omega} d^n \xi),$$

which are localized in a small neighborhood of the point ξ_0 in the sense that they satisfy the condition

$$\frac{(\psi, o(\xi - \xi_0)\psi)}{(\psi, \hat{H}_0^{(DW)}\psi)} \ll 1, \tag{37}$$

where $o(\xi - \xi_0)$ is a residual term in the right-hand side of (34). Thus, the approach exposed which is relevant to call the quasiclassical one gives, in the mentioned approximate sense, a unique and diffeoinvariant Hamilton operator.

At the same time, DeWitt and some other authors considered the appearance of the potential $(\hbar^2/12m)R_{(\omega)}(\xi)$ as an unfavorable phenomenon because traditionally $\hat{H}_0^{(\text{kin})}$ was taken as the Hamiltonian of the particle in V_n . They added an appropriate counterterm into the Lagrangian, that is into the integrand in formula (32), to have $\hat{H}_0^{(\text{kin})}$ instead of $\hat{H}_0^{(\text{DW})}$ in Eq. (34). However, the corrected Lagrangian is not the one of geodesic motion of which quantization is the matter of the present paper. Actually, the appearance of $R_{(\omega)}$ in the Hamiltonian is quite in conformity with the QFT-approach (Tagirov, 1999) a brief exposition of which will be given in Section 6.

3.2. Comparison of Canonical and Dewitt's Hamiltonians

Now, let us compare the Hamiltonian $\hat{H}_0^{(v)}$, obtained exactly in the canonical sense and the approximate quasiclassical one $\hat{H}_0^{(\text{DW})}$. Remind that the latter was obtained by retracting the point ξ to ξ_0 along a geodesic line connecting them. Thus, a position of ξ with respect to ξ_0 is naturally defined by the geodesic distance $s(\xi, \xi_0)$ between them and the tangent vector $(d\xi^i/ds)_0$ to the geodesic line at ξ_0 . These quantities form *the Riemannian coordinates*

$$y^i(\xi) \stackrel{\text{def}}{=} s(\xi, \xi_0) \left(\frac{d\xi^i}{ds} \right)_0 \tag{38}$$

with the origin at the point ξ_0 . In these coordinates the metric tensor ω_{ij} , its derivatives by y^i and, respectively, the Christoffel symbols γ_{kl}^i are represented as a power series in y^i , coefficients of which are polynomials in powers of the components of the Riemann–Christoffel tensor and of its covariant derivatives taken at the origin $y^i = 0$, i.e., at ξ_0 . Therefore, applying the Veblen method of affine extensions (Veblen, 1927) using contracted Bianchi identities, one can represent the quantum potential $V_q^{(v)}$ as a similar series. For our discussion, the following two terms of the series are sufficient:

$$\hat{H}_0^{(v)}(y) = -\frac{\hbar^2}{2m} \left(\Delta_{(\omega)}(y) - \frac{\nu}{12} R_{(\omega)} \Big|_{y=0} - \frac{\nu}{12} (\partial_i R_{(\omega)}) \Big|_{y=0} y^i + O((y)^2) \right). \tag{39}$$

The condition of coincidence of $\hat{H}_0^{(v)}(y)$ with $\hat{H}_0^{(\text{DW})}$ in the zero-order approximation is satisfied for the value $\nu = 2$ in (23) and (25). Thus, from the canonical point of view adopted here, *the correct nonrelativistic Hamilton operator* for a point-like particle in the globally static $V_{1,n}$ is the following remarkably simple expression:

$$\hat{H}_0^{(2)} = \frac{1}{2m} \hat{p}_i \omega^{ij}(\hat{\xi}) \hat{p}_j. \tag{40}$$

This solves the ambiguity problem of ordering of the primary operators in the canonical quantization of the geodesic motion. However, the problem of diffeon-invariance of quantum potential $V_q^{(2)}$ retains. This problem, as well as the problem of ordering for the Hamiltonians which are not quadratic in momenta, will be discussed in Section 5. And now we pass to a justification of the obtained result coming from consideration of another traditional approach to formulation of quantum mechanics.

4. QUANTIZATION OF GEODESIC MOTION BY PATH INTEGRATION

4.1. Relation Between Canonical and Path Integration Formalisms

Not only the point-like particle motion but also a number of other mechanical problems are naturally represented as a geodesic motion or its generalization in some V_n . Usually, the latter are homogeneous spaces of symmetry groups (see, for instance, Groshe *et al.*, 1997; Marinov, 1995; and references therein). For this class of systems, the Feynman formalism of path integrals is considered as a very appropriate approach to solve the Schrödinger equation for the particle propagator since it takes into account the metric of the configuration space through a natural measure and representation of the virtual path as consisting of small segments of geodesic lines.

In this approach, the path integral relates a given quantum Hamiltonian \hat{H}_0 represented as a differential operator in $L^2(V_n; \mathbb{C}; \sqrt{\omega}d^n\xi)$ to some effective classical Lagrangian (Marinov, 1995). The Hamiltonian may be considered as a result of quantization of the classical dynamics described by the Lagrangian so found. An inverse problem can be posed: to select $\hat{H}_0 \stackrel{def}{=} \hat{H}_0^{(F)}$ (the superscript (F) denotes ‘‘Feynman’’ as will be clear a bit below) so that the effective Lagrangian would prove to be the classical one for the geodesic motion:

$$L_{\text{eff}}(\xi, \dot{\xi}) = L_{\text{cl}}(\xi, \dot{\xi}) \equiv \frac{m}{2} \omega_{ij}(\xi) \dot{\xi}^i \dot{\xi}^j. \tag{41}$$

A correspondence $H_0 \rightarrow \hat{H}_0^{(F)}$ thus defined and taken together with the map (16) of the primary observables may be called *the Feynman quantization* of the geodesic motion in V_n . Consider such an approach in a brief descriptive form sufficient for a comparison with the formalisms considered earlier.

So, a problem is to represent, as a path integral, the following formal propagator in V_n

$$\mathbb{K}(\xi'', t'' | \xi', t') = \langle \xi'' | e^{-\frac{i}{\hbar}(t''-t')\hat{H}_0} | \xi' \rangle, \tag{42}$$

for the quantum Hamiltonian of the form

$$\hat{H}_0 = -\frac{\hbar^2}{2m} \Delta_\omega(\xi) + V(\xi), \tag{43}$$

acting in $L^2(V_n; \mathbb{C}; \sqrt{\omega} d^n \xi)$. Here we consider as an already-established fact that the set of the possible (nonrelativistic) Feynman Hamiltonians $\hat{H}_0^{(F)}$ a particle in V_n is contained among Hamiltonians (43) with arbitrary potentials $V(\xi)$.

Following the line of calculations by D’Olivo and Torres (1989), divide the time interval $[t', t'']$ by $N \rightarrow \infty$ intervals of infinitesimal duration $\epsilon = (t'' - t')/N$ and represent $\mathbb{K}(\xi'', t'' | \xi', t')$ as follows:

$$\mathbb{K}(\xi'', t'' | \xi', t') = \lim_{n \rightarrow \infty} \int \prod_{l=1}^{N-1} \sqrt{\omega(\xi_l)} d^n \xi_l \prod_{j=1}^{N-1} \langle \xi_l | e^{-\frac{i}{\hbar} \epsilon \hat{H}_0} | \xi_j \rangle, \quad (44)$$

where $\xi_0 = \xi', \xi_N = \xi''$.

To calculate the matrix elements of \hat{H}_0 in the configuration representation, one should represent the differential operator $\Delta_\omega(\xi)$ in (43) through $\hat{\xi}, \hat{p}$. To this end, a *rule of ordering of them* should be fixed. Contrary to D’Olivo and Torres (1989), who, as many other authors on the matter, adopted the Weyl rule, we use a more general rule (23). Then, we have

$$\hat{H}_0 = \hat{H}_0^{(v)} - V_q^{(v)}(\xi) + V(\xi), \quad (45)$$

where $\hat{H}_0^{(v)}$ and $V_q^{(v)}(\xi)$ are assumed to be expressions (25) and (27) respectively. Calculation of the matrix elements within the terms linear in ϵ using our generalized rule of ordering gives

$$\begin{aligned} \mathbb{K}(\xi'', t'' | \xi', t') &= \lim_{N \rightarrow \infty} \int \left(\frac{1}{2\pi i \hbar \epsilon} \right)^{\pi N/2} \prod_{l=1}^{N-1} \sqrt{\omega(\xi_l)} d^n \xi_l \\ &\times \prod_{j=1}^{N-1} \left\{ \frac{(\sqrt{\tilde{\omega}})^{(v)}(\xi_{j-1}, \xi_j)}{[\omega(\xi_j)\omega(\xi_{j-1})]^{1/4}} \exp \left\{ \frac{i}{\hbar} \epsilon \tilde{L}_{\text{eff}}^{(v)} \left(\xi_{j-1}, \xi_j; \frac{\Delta \xi_j}{\epsilon} \right) \right\} \right\}, \\ \Delta \xi_j &\equiv \left\{ \Delta \xi_j^i \stackrel{\text{def}}{=} \xi_j^i - \xi_{j-1}^i \right\}. \end{aligned} \quad (46)$$

Here $(\sqrt{\tilde{\omega}})^{(v)}(\xi_{j-1}, \xi_j)$ and $\tilde{L}_{\text{eff}}^{(v)}(\xi_{j-1}, \xi_j, \Delta \xi_j/\epsilon)$ are the quantities that are expressed, respectively, through the functions $\sqrt{\omega}(\xi)$ and

$$L_{\text{eff}}^{(v)} \left(\xi, \frac{\Delta \xi_j}{\epsilon} \right) \stackrel{\text{def}}{=} L_{\text{cl}}(\xi, \Delta \xi_j/\epsilon) - V(\xi) + V_q^{(v)} \quad (47)$$

along the following general rule implied by Eq. (23):

$$\tilde{f}^{(v)}(\xi_{j-1}, \xi_j) = v f(\bar{\xi}_j) + \frac{1-v}{2} (f(\xi_{j-1}) + f(\xi_j)), \quad \bar{\xi}_j \stackrel{\text{def}}{=} \frac{1}{2} (\xi_j + \xi_{j-1}). \quad (48)$$

Now, the product in J in Eq. (46) should be represented as a product of exponentials of some classical action on the intervals $[\xi_{j-1}, \xi_j]$, that is as a product

of factors of the form

$$\exp \left\{ \frac{i}{\hbar} \epsilon L'_{\text{eff}}(\xi'_J, \Delta \xi_J / \epsilon) \right\}, \tag{49}$$

where, in the exponent, the value of some effective Lagrangian $L'_{\text{eff}}(\xi, \dot{\xi})$ (in general, it differs from $L^{(v)}_{\text{eff}}$) stands, which is taken at the point $\xi'_J \in [\xi_{J-1}, \xi_J]$ remained arbitrary for a time being.

To obtain the representation, all functions of $\xi_{J-1}, \xi_J, \bar{\xi}_J$ under the product in J should be expanded into the Tailor series near the point ξ'_J up to terms quadratic in $\Delta \xi_J$, since only such terms contribute to the integral Eq. (46). Further, one should include the contribution of the preexponential factor to the exponent in a form of an additional quantum potential. Consider this procedure separately for the two principally different cases.

- (A) The intermediate point evaluation of the integrands: $\xi'_J = (1 - \mu)\xi_{J-1} + \mu\xi_J, 0 < \mu < 1$, i.e., $\xi'_J \in (\xi_{j-1}, \xi_j)$.
- (B) The endpoint evaluation of the integrands: $\xi' = \xi_{J-1}$ or $\xi' = \xi_J$, i.e., ξ' is taken at the ends of the closed interval $[\xi_{J-1}, \xi_J]$.

4.2. Quantum Potential for the Intermediate Point Evaluation of Integrands (Case A)

For the generic function (48), one has

$$\begin{aligned} \tilde{f}^{(v)}(\xi_{J-1}, \xi_J) &= f(\xi'_J) + \left(\frac{1}{2} - \mu \right) \partial_i f(\xi'_J) \Delta \xi^i_j \\ &+ \frac{1}{2} \left(\frac{2-v}{4} - \mu + \mu^2 \right) \partial_i \partial_j f(\xi'_J) \Delta \xi^i_j \Delta \xi^k_j. \end{aligned} \tag{50}$$

Apply this general formula to $f(\xi) \equiv L^{(v)}_{\text{eff}}(\xi, \Delta \xi / \epsilon)$. The last term in Eq. (50) turns out to be equal to

$$\frac{1}{2} \left(\frac{2-v}{4} - \mu + \mu^2 \right) \partial_i \partial_j \omega_{kl}(\xi'_J) \Delta \xi^j_i \Delta \xi^k_j \frac{\Delta \xi^l_j}{\epsilon} \frac{\Delta \xi^k_j}{\epsilon} \tag{51}$$

in the necessary order of ϵ .

Further, we use the result by McLaughlin and Schulman (1971) according to which the following substitution can be made under the integration in Eq. (46):

$$\Delta \xi^i_j \Delta \xi^j_j \rightarrow i \epsilon \frac{\hbar}{m} \omega^{ij}(\xi'_J). \tag{52}$$

After this substitution in Eq. (51) and symmetrization of the resulting expression in indexes i, j, k, l , one comes to the quantum potential

$$V_L^{(v;\mu)}(\xi'_j; v; \mu) = -\frac{\hbar^2}{12m} \left(\frac{2-v}{4} - \mu + \mu^2 \right) (\partial_i \partial_j \omega_{kl} (\omega^{ij} \omega^{kl} + 2\omega^{ik} \omega^{jl})) (\xi'_j), \tag{53}$$

in addition to $L_{\text{eff}}^{(v)}(\xi'_j, \Delta \xi_j / \epsilon)$. Another additional term here,

$$i \left(\frac{1}{2} - \mu \right) \frac{\hbar}{3} \partial_i \omega_{kl} (\xi'_j) \left(\omega^{kl} (\xi'_j) \frac{\Delta \xi_j^i}{\epsilon} + 2\omega^{ik} (\xi'_j) \frac{\Delta \xi_j^l}{\epsilon} \right), \tag{54}$$

comes from the second term in Eq. (50) after the use of the same substitution (52). It adds to $L_{\text{eff}}^{(v)}$, a term which is proportional to $\Delta \xi^i / \epsilon \sim \dot{\xi}^i$ that is linear in the velocity. There is no such term in L_{cl} and there is nothing to compensate it so that the condition (41) were satisfied. Indeed, the logarithm of the preexponential factor

$$\tilde{\Omega}_J = \frac{(\sqrt{\tilde{\omega}})^{(v)}(\xi_{J-1}, \xi_J)}{[\omega(\xi_J)\omega(\xi_{J-1})]^{1/4}} \tag{55}$$

does not contain a term linear in $\epsilon \dot{\xi}$: when $\epsilon \rightarrow 0$:

$$\begin{aligned} \tilde{\Omega}_J &= 1 - \left(\frac{v}{8} \partial_i \partial_j \ln \omega(\xi'_j) - \left(\frac{3-v}{32} - \frac{\mu}{4} + \frac{\mu^2}{4} \right) \partial_i \ln \omega(\xi'_j) \partial_j \ln \omega(\xi'_j) \right) \Delta \xi^i \Delta \xi^j \\ &+ O((\Delta \xi)^2) \equiv \Omega(\xi'_j, v; \mu). \end{aligned} \tag{56}$$

Therefore, to avoid appearance of a term proportional to the velocity in L_{eff} , one should take

$$\mu = \frac{1}{2}, \tag{57}$$

i.e., $\xi'_j = \bar{\xi}_j$, as it is taken by D’Olivo and Torres (1989), who adopt the Weyl ordering (formula (48) for $v = 1$) from the beginning.

Taking into account the condition (57) and the substitution (52), one can reduce the contribution of $\tilde{\Omega}$ into the path integral (46) to that one more quantum potential $V_{\Omega}^{(v)}$ is added to $L_{\text{eff}}^{(v)}$ in the exponent of the exponential:

$$V_{\Omega}^{(v)}(\xi) = -\frac{\hbar^2}{2m} \omega^{ij}(\xi) \left(\frac{v}{8} \partial_i \partial_j \ln \omega(\xi) - \left(\frac{1-v}{32} \right) \partial_i \ln \omega(\xi) \partial_j \ln \omega(\xi) \right) + O(\epsilon^2). \tag{58}$$

As a result, if one chooses in the initial formula (43)

$$\begin{aligned} V(\xi) &\equiv V_A^{(F;v)}(\xi) \stackrel{\text{def}}{=} V_q(\xi) + V_L^{(v)}(\xi) + V_{\Omega}^{(v)}(\xi) \\ &= -\frac{\hbar^2}{24m} \left(\frac{2v+7}{2} \omega^{ij} \omega^{kl} - (5-2v) \omega^{ik} \omega^{jl} \right) \partial_i \partial_j \omega_{kl} \end{aligned}$$

$$\begin{aligned}
 & + \frac{\hbar^2}{4m} \left(\frac{\nu + 2}{4} \omega^{km} \omega^{ln} \omega^{ij} - \frac{\nu - 2}{4} \omega^{im} \omega^{jn} \omega^{kl} - (\nu - 2) \omega^{im} \omega^{kn} \omega^{jl} \right) \\
 & \times \partial_i \omega_{mm} \partial_j \omega_{kl}, \tag{59}
 \end{aligned}$$

then, in the integrand of the path integral, only the following product remains in the required approximation

$$\prod_{j=1}^{N-1} \exp \left\{ \frac{i\epsilon}{\hbar} L_{cl}(\bar{\xi}_j) \right\}, \tag{60}$$

that is a product of exponentials of the ratio of the classical action of the geodesic motion between the points ξ_{j-1} and ξ_j to the Planck constant \hbar .

Thus, we have determined a map $H_0 \rightarrow H_{0A}^{(F;\nu)}$ of the Hamilton function of the geodesic motion in V_n on operator (43) with quantum potential (59) is a version acting on $L^2(V_n; \mathbb{C}; \sqrt{\omega} d^n \xi)$ is a version of the Feynman quantization of the geodesic motion in V_n . It is not diffeoinvariant as well as contains freedom in the choice of the value of parameter ν corresponding to arbitrariness of the ordering rule in the canonical quantization. Could one select ν so that $V_A^{(F;\nu)}(\xi)$ would coincide with the result of the quasiclassical quantization (36) in the region where such comparison is relevant, i.e., in a neighborhood of the origin of the normal Riemannian coordinates y^i ? The answer is no, it is not possible because

$$V_A^{(F;\nu)}(y) = \frac{\hbar^2}{2m} \frac{R}{3}(0) + O(y) \tag{61}$$

independent of the value of ν and, actually, independent on the choice of μ . Thus, the initial ambiguity of the canonical quantization not only retains but also becomes larger in the considered version of the Feynman quantization.

4.3. Quantum Potential for the Endpoint Evaluation of Integrands (Case B)

In this case, if one takes $\mu = 0$ or $\mu = 1$, again the inadmissible addition of a term linear in ξ to the exponent of the exponential occurs. It is a consequence of an asymmetric contribution of the endpoints of the interval $[\xi_{j-1}, \xi_j]$ while, for a given function $f(\xi)$, expression (48) for $\tilde{f}^{(\nu)}(\xi_{j-1}, \xi_j)$ depends on the endpoints symmetrically. However, the following symmetric expression for $\tilde{f}^{(\nu)}$,

$$\begin{aligned}
 \tilde{f}^{(\nu)}(\xi_{j-1}, \xi_j) & = \frac{1}{2} f(\xi_{j-1}) + \frac{1}{2} f(\xi_j) + \frac{\nu}{8} (\partial_i f(\xi_{j-1}) - \partial_i f(\xi_j)) \Delta \xi_j^i \\
 & + \frac{\nu}{16} (\partial_i \partial_j f(\xi_{j-1}) + \partial_i \partial_j f(\xi_j)) \Delta \xi_j^i \Delta \xi_j^j + O((\Delta \xi_j)^3), \tag{62}
 \end{aligned}$$

can be easily obtained if ξ_{J-1} and ξ_J are at a short distance. Applying this formula to $\tilde{f}^{(v)} \equiv \tilde{L}_{\text{eff}}^{(v)}$ in the exponent in formula (46), one should consider contributions of the adjacent intervals $[\xi_{J-2}, \xi_{J-1}]$ and $[\xi_J, \xi_{J+1}]$ at the points ξ_{J-1} and ξ_J , respectively.

The total contribution to the phase at ξ_J of the terms of $\tilde{L}_{\text{eff}}^{(v)}$, which are linear in $\Delta\xi$, is

$$i \frac{v\hbar}{8m} \epsilon \omega^{kl}(\xi_J) \partial_i \omega_{kl}(\xi_J) \ddot{\xi}^i, \quad (63)$$

and it can be neglected in the path integration. Here, substitution (52) and relation

$$\Delta\xi_{J-1} = \Delta\xi_J - \epsilon^2 \ddot{\xi}_J + O(\epsilon^3) \quad (64)$$

are used. Making these substitutions in the terms which are quadratic in $\Delta\xi$, one obtains that a quantum potential

$$V_B^{(v)} = -\frac{v\hbar^2}{24m} ((\omega^{ij}\omega^{kl} + 2\omega^{ik}\omega^{jl}) \partial_i \partial_j \omega_{kl}) \quad (65)$$

is added to $\tilde{L}_{\text{eff}}^{(v)}$.

The contribution to the phase of the adjacent preexponential terms $\tilde{\Omega}_J$ and $\tilde{\Omega}_{J+1}$,

$$\tilde{\Omega}_J \cdot \tilde{\Omega}_{J+1} = \exp(\ln \tilde{\Omega}_J + \ln \tilde{\Omega}_{J+1}), \quad (66)$$

can be calculated in a similar way. To this end, expand the terms in the exponents in powers of $\Delta\xi_J$ and $\Delta\xi_{J+1}$ up to $O((\Delta\xi)^3)$ and collect the terms with the coefficients that depend on ξ_J . The remaining terms go over to the analogous contributions at the points ξ_{J-1} and ξ_{J+1} . Then, using relation (64), one obtains the following function of ξ_J :

$$\frac{v-2}{8} \epsilon^2 \ddot{\xi}_{J+1}^i \partial_i \ln \omega + \left(\frac{2-v}{16} \partial_i \partial_j \ln \omega + \frac{1}{32} \partial_i \ln \omega \partial_j \ln \omega \right) \cdot \Delta\xi_J^i \Delta\xi_J^i + O(\epsilon^2 \Delta\xi). \quad (67)$$

Obviously, the first term here can be neglected under the integration. Hence, using substitution (52), one finds a contribution to the phase at the point ξ_J in a form of the following quantum potential:

$$V_\Omega^{(v)} = -\frac{\hbar^2}{32m} \omega^{ij} (2(2-v) \partial_i \partial_j \ln \omega + \partial_i \ln \omega \partial_j \ln \omega). \quad (68)$$

Then, one should put

$$V(\xi) \equiv V_B^{(F,v)}(\xi) \stackrel{\text{def}}{=} V_q^{(v)}(\xi) + V_L(\xi) + V_\Omega^{(v)}(\xi) \quad (69)$$

in (43) to retain in the phase only a ratio of the classical action near the point ξ_J for the time interval ϵ to the Plank constant \hbar .

Now, let us consider $V_B^{(F;\nu)}$ at the origin of the normal Riemannian coordinates y^j . Note at once that $V_L^{(F;\nu)}(y) = O(y)$ since

$$\partial_i \partial_j \omega_{kl}(y) = \frac{1}{3}(R_{(\omega)ikjl} + R_{(\omega)iljk})(0) + O(y), \tag{70}$$

(see, for instance, Synge, 1960). A nonvanishing contribution into $V_\Omega^{(\nu)}$ can be given only by the first term in (68). The contribution vanishes identically if and only if

$$\nu = 2. \tag{71}$$

Thus, we come to a remarkable justification of the ordering rule which had been found by comparison of the canonical and quasiclassical Hamiltonians in Section 3. At the same time, we have fixed a unique way to calculate the path integral and, in particular, a prescription to evaluate the integrand functions: *they should be evaluated at the nodes of the lattice of integration*. The prescription differs from that induced by the Weyl ordering according to which the evaluation should be done in the midpoints of intervals of the lattice.

It should be noticed also that the ordering corresponding to $\nu = 2$ was mentioned among many other ones by D’Olivo and Torres (1989), but we have singled out it from a two-parametric (in ν and μ) set of possible ordering with a necessity. In the next section, a question will be discussed in particular why the comparison of quantum Hamiltonians in a vicinity of the origin of the Riemannian coordinates has a special geometric meaning. As for now, we give the complete expression for $V_B^{(2)}$:

$$\begin{aligned} V_B^{(F;2)} = & -\frac{\hbar^2}{12m}(2\omega^{ij}\omega^{kl} + \omega^{ik}\omega^{jl})\partial_i\partial_j\omega_{kl} \\ & -\frac{\hbar^2}{16m}(2\partial_i\omega^{ij}\partial_j\ln\omega + \omega^{ij}\partial_i\ln\omega\partial_j\ln\omega). \end{aligned} \tag{72}$$

Of course, this Feynman quantum potential differs, in general, from the canonical one:

$$V_q^{(2)}(\xi) = -\frac{\hbar^2}{4m}\left(\partial_i(\omega^{ij}\gamma_j) + \partial_i\partial_j\omega^{ij} - \frac{1}{2}\omega^{ij}\gamma_i\gamma_j\right), \tag{73}$$

(i.e., Eq. (27) for $\nu = 2$) and the question remains, which of the potentials is “more correct?”

5. DISCUSSION OF THE RESULTS OBTAINED

Thus, taking $\nu = 2$ in Eq. (23) is proposed as a concrete and unambiguous solution of the problem of arbitrariness the ordering rule, one of the main difficulties

of the canonical quantum mechanics in V_n . However, the rule is obtained namely for observables (Hamiltonians) which are quadratic in momenta. If one attempts to adopt the logic of our construction for an observable of a more complicated structure, the rule of ordering thus obtained will determine its own “bracket” in the quantization condition (Q2). It is unclear, will this rule be unambiguous but, in any case, we come to the conclusion that *for different classes of observables, there should be used different “brackets”* $\{.,.\}_\hbar$ in condition (Q2). This conclusion may seem rather strange, but, at least, it does not contradict to the known experimental data since the corrections to the Poisson bracket in the left-hand side of condition (Q2) are very small and, correspondingly, differences of corrections for different versions are small too.

Further, the result refers to the nonrelativistic version of the geodesic dynamics. A more difficult problem of quantization of the relativistic version remains, which is based on Hamiltonian $H(\xi, p)$, eq. (7). Its possible asymptotic solution by the use of Von Neumann’s rule in the form of (11) has already been given in Section 2. However, if, for any classical Hamiltonian, its own canonical quantization has to be constructed, then the way which was followed for the Hamiltonian (6) should be passed anew for (7). In this case, an analog of the quasiclassical Hamiltonian $\hat{H}_0^{(DW)}$ should apparently be the quantum Hamiltonian calculated in the Blattner–Costant–Souriau formalism for the terms of the asymptotic expansion (7). It is calculated for the first four terms by Kalinin (1999) and differs from the result of an immediate application of the Von Neumann rule (11). An analysis of this difference seems to be an interesting task for understanding relations between different formalisms of quantization.

Let us pass now to the important point that, to determine the rule $\nu = 2$, it was principal to compare the Hamiltonians $\hat{H}_0^{(\nu)}$, $\hat{H}_0^{(DW)}$, $\hat{H}_{0A}^{(F;\nu)}$, and $\hat{H}_{0B}^{(F;\nu)}$ in a vicinity of the origin of the Riemannian coordinates y^a namely. Why is this system distinguished among all possible systems? An answer is apparently as follows. The position of a point $\{\xi^i\}$ is defined in the Riemannian system completely by the geodesic line connecting the point with the origin $\{\xi_0^i\}$ and, therefore, only by the metric of V_n . Indeed, according to Eq. (38) the normal Riemannian coordinates $y^a(\xi) = n^{(a)}(\xi)s(\xi; \xi_0)$ of the point $\{\xi^i\}$ are completely defined by values of the geodesic distance $s(\xi; \xi_0)$ and projections $n^{(a)}(\xi) = e_i^{(a)}(d\xi^i/ds)_0$ of the tangent vector to the geodesic line connecting ξ and ξ_0 . The coordinate lines

$$y^1 = \text{const}, \dots, y^{k-1} = \text{const}, y^{k+1} = \text{const}, \dots, y^n = \text{const}, \quad 1 \geq k \leq n, \tag{74}$$

are distinguished by that their all n curvatures vanish. Imagine that a similar system of coordinates realized not by the geodesics but by the lines determined by some other equation. Take, for example, the geodesic equation with an external force in the right-hand side. Such line has, at least, one proper curvature determined by the force (see a physical oriented exposition of the question by Synge, 1960).

Respectively, these exterior fields of curvatures of the coordinate lines enter into quantum theory.

Thus, the class of the Riemannian coordinates turns out to be a preferred one. It seems to contradict the dogma of general relativity on equivalence of possible systems of coordinates. The contradiction may possibly be solved as follows. A quantum–mechanical description of a physical system should include an indication of the way of measurement (observation) of properties of the system (for a recent discussion of the question, see Rovelli, 1996). In the Schrödinger representation, a system of coordinates $\{\xi^i\}$ plays two roles simultaneously. On the one hand, it arithmetizes (“digitizes”) the configuration space by its local map on R_n . On the other hand, it specifies n primary observables represented in quantum mechanics by the operators $\hat{\xi}$ the spectra of which may be considered as formalization of indications of an apparatus detecting a position of the particle. Numerical values of the indications should not depend on the arithmetization of V_n and, in this sense, should be represented by scalars with respect to transformations of ξ^i ’s. Therefore, let us separate the two roles of the coordinates as follows: keep for the arbitrary coordinates ξ^i the role of arithmetization of V_n and introduce $2n$ canonically conjugate scalar functions $q^{(i)}(\xi), p_{(j)}(\xi, p)$ by the following canonical transformation:

$$\{\xi^i, p_j\} \rightarrow \{q^{(k)}(\xi), p_{(l)}(\xi, p)\}. \tag{75}$$

Here $q^{(k)}(\xi)$, are fixed $2n$ functions such that $\text{rank } \|\partial_i q^{(j)}\| = n$, and $p_{(l)}(\xi, p) \stackrel{\text{def}}{=} K_{(l)}^i(\xi)p_i$ where

$$K_{(j)}^i(\xi) = \frac{1}{2} \det \|\partial_k q^{(l)}\| \epsilon^{ii_2 \dots i_n} \epsilon_{(jj_2 \dots j_n)} \partial_{i_2} q^{(j_2)} \dots \partial_{i_n} q^{(j_n)} \tag{76}$$

are n vector fields and $\epsilon^{i_1 i_2 \dots i_n}, \epsilon_{(j_1 j_2 \dots j_n)}$ are completely antisymmetric symbols for both upper and lower indices. Of course, one may take $q^{(i)}(\xi) \equiv \xi^i$ as a particular case, which means that the arithmetization of V_n and observation of the particle position are performed by the same tools.

The operators in $L^2(V_n; \mathbb{C}; \sqrt{\omega} d^n \xi)$, corresponding to the scalar primary observables $q^{(i)}(\xi), p_{(j)}(\xi)$ are

$$\hat{q}^{(i)}(\xi) = q^{(i)}(\xi) \cdot \hat{\mathbf{1}}, \tag{77}$$

$$\hat{p}_{(j)} = -i\hbar \left(K_{(j)}^l(\xi) \nabla_l + \frac{1}{2} \nabla_l K_{(j)}^l(\xi) \right). \tag{78}$$

Introduce a scalar Hamilton operator $\hat{H}_0^{(v)'}(\xi)$ from the condition that it coincides with $\hat{H}_0^{(v)}(\xi)$ when $q^{(i)}(\xi) \equiv \xi^i$. Restricting for brevity to the case of $v = 2$, one has

$$\hat{H}_0^{(2)'} \stackrel{\text{def}}{=} \frac{1}{2m} \hat{p}_{(i)} \partial_k q^{(i)} \omega^{kl} \partial_l q^{(j)} \hat{p}_{(j)} = -\frac{\hbar^2}{2m} \left(\Delta_{(\omega)} - \frac{1}{2} \nabla^k v_k + \frac{1}{4} v^k v_k \right), \tag{79}$$

$$v_k \stackrel{def}{=} K_{(i)}^m \nabla_m \partial_k q^{(i)}. \quad (80)$$

The quantum potential in the right-hand side of (79) does not depend on the choice of coordinates ξ^i , but does on the choice of the observables of position $q^i(\xi)$. This corresponds to the concept of *relational quantum mechanics* developed by Rovelli (1996) according to which different methods of observation of a quantum system give different amounts of information on the system. One may think that choosing the Riemannian coordinates y^a as observables, i.e., $q^{(a)}(\xi) \equiv y^a(\xi)$ gives maximal information on the quantum analogue of the particle moving along a geodesic line in V_n because, in this case, no outside information is added in the form of the proper curvatures of coordinate lines.

6. ON QFT-APPROACH TO QUANTUM MECHANICS IN CURVED SPACES

6.1. Quantum Field Theoretical Basis

To give a more complete exposition of the problem of quantum mechanics in V_n , an approach which is an alternative to quantization of the geodesic motion will be outlined in the present section (details can be found in Tagirov, 1999). It was mentioned in Section 1 as the QFT-approach.

The approach is based on quantum theory of the linear real scalar field $\varphi(x)$, $x \in V_{1,n}$ in $V_{1,n}$, of which the quanta are supposed to be the point-like spinless and chargeless particles. Thus, one may think that it should have a domain of intersection with the approaches based on quantization of the geodesic motion and considered in the preceding sections.

A sufficiently general equation of the field in $V_{1,n}$ is

$$\square \varphi + \zeta R_{(g)}(x) \varphi + \left(\frac{mc}{\hbar} \right)^2 \varphi = 0, \quad x \in V_{1,n}, \quad \square \stackrel{def}{=} g^{\alpha\beta} \nabla_\alpha \nabla_\beta. \quad (81)$$

Here ζ is an arbitrary dimensionless real constant, $R_{(g)}(x)$ and ∇_α are, respectively, the scalar curvature and the covariant derivative in $V_{1,n}$. Two values of ζ are especially distinguished. For $\zeta = 0$, the field $\varphi(x)$ interacts with the external gravitational field $g_{\alpha\beta}(x)$ minimally, that is switched on by immediate substitution of the partial derivatives with respect to the Cartesian coordinates in the standard Klein–Gordon equation by the covariant derivatives with respect to $V_{1,n}$. For $\zeta = (n-1)/4n$, the interaction is conformal-invariant in the limit of $m=0$; for $n=3$ this property was first noticed by Penrose (1963) and studied in detail by Chernikov and Tagirov (1968) and Tagirov (1973). The latter authors had found some other properties of the theory with the conformal coupling, which are favorable from the physical point of view.

In the globally static $V_{1,n}$, by which the scope of this paper is restricted, one has $R_{(g)} = -R_{(\omega)}$. If, in addition, $n = 3$, and $\zeta = 1/6$ then a nonrelativistic limit of (81) for conformal coupling will be the Schrödinger equation just with the Hamiltonian $\hat{H}_0^{(DW)}$. However, an almost metaphysical question arises here: Why the quasiclassical approximation leads to the Hamiltonian $\hat{H}_0^{(DW)}$ to $\zeta = 1/6$ for any dimension n while the scalar field theory with the conformal coupling leads to the same Hamiltonian only for the dimension of the real world $n = 3$?

For a time being, we consider again the general metric (1), not necessarily the globally static one. Canonical quantization of $\varphi(x)$ in the general $V_{1,n}$ in the Fock representation is essentially based on the complexification $\Phi_c = \Phi \otimes \mathbb{C}$ of space Φ of solutions to Eq. (81) and a selection of $\Phi'_c \subset \Phi_c$ which can be represented as

$$\Phi'_c = \Phi^- \oplus \Phi^+ \tag{82}$$

(see, for instance, Gibbons and Pohle, 1993). Here, Φ^- and Φ^+ are mutually complex conjugate subspaces of Φ_c , for which the conserved sesquilinear form

$$\{\varphi_1, \varphi_2\}_\Sigma \stackrel{def}{=} i \int_\Sigma d\sigma^\alpha (\overline{\varphi_1}(x) \partial_\alpha \varphi_2(x) - \partial_\alpha \overline{\varphi_1}(x) \varphi_2(x)), \tag{83}$$

is respectively positive and negative, and thus provides Φ^- and Φ^+ with pre-Hilbert structures.

Assume that a formal (and auxiliary) basis $\{\varphi(x; A)\}$ in Φ^- exists, which is enumerated by a multi-index A having values on a set $\{A\}$ with a measure $\mu(A)$ and orthonormalized with respect to the inner product (83). Then, the quantum field operator in a Fock space \mathcal{F} can be introduced

$$\check{\varphi}(x) = \int_{\{A\}} d\mu(A) (\check{c}^+(A) \overline{\varphi}(x; A) + \check{c}^-(A) \varphi(x; A)) \equiv \check{\varphi}^+(x) + \check{\varphi}^-(x). \tag{84}$$

(Here and further, operators in \mathcal{F} are denoted as \check{O} and called *QFT-operators* contrary to the quantum-mechanical ones, or *QM-operators*, which are denoted throughout the paper as \hat{O} .) The operators $\check{c}^+(A)$ and $\check{c}^-(A)$ are creation and annihilation of the field modes $\varphi^-(x; A) \in \Phi^-$ (or, of *quasiparticles*). They satisfy the canonical commutation relations

$$[\check{c}^+(A), \check{c}^+(A')] = [\check{c}^-(A'), \check{c}^-(A')] = 0, \\ \int_{\{A\}} d\mu(A) f(A) [\check{c}^-(A), \check{c}^+(A')] = f(A'),$$

for any appropriate function $f(A)$. They act in the Fock space \mathcal{F} with the cyclic vector $|0\rangle$ (*the quasivacuum*) defined by the equations

$$\check{c}^-(A)|0\rangle = 0. \tag{85}$$

6.2. QFT-Operators of Observables

Now, the following diffeoinvariant quantum field observables can be naturally introduced. *The QFT-operator of a number of quasiparticles* is determined in the standard way (see, for instance, Schweber, 1961, chap. 7, Section 3):

$$\begin{aligned} \check{N}(\check{\varphi}; \Sigma) &\stackrel{def}{=} \int_{\Sigma} d\sigma^{\alpha} (\check{\varphi}^+ \partial_{\alpha} \check{\varphi}^- - \partial_{\alpha} \check{\varphi}^+ \check{\varphi}^-), \\ &\stackrel{def}{=} \int_{\Sigma} d\sigma(x) \check{N}(x), \end{aligned} \tag{86}$$

where $d\sigma(x) \stackrel{def}{=} \sqrt{\omega(t, \xi)} d^n \xi$ is the inner volume element of Σ .

The QFT-operator of the projection of momentum of the field $\hat{\varphi}(x)$ on a given vector field $K^{\alpha}(x)$ is also a standard expression determined by the general-relativistic Lagrangian for φ :

$$\check{P}_K(\check{\varphi}; \Sigma) \stackrel{def}{=} : \int_{\Sigma} d\sigma^{\alpha} K^{\beta} T_{\alpha\beta}(\check{\varphi}) :, \tag{87}$$

where the colons denote the normal product of operators $\check{c}^{\pm}(A)$ and $T_{\alpha\beta}(\varphi)$ is the metric energy–momentum tensor of the field $\varphi(x)$.

The n QFT-operators

$$\begin{aligned} \check{Q}^{(i)}\{\check{\varphi}; \Sigma\} &= i \int_{\Sigma} d\sigma^{\alpha}(x) q_{\Sigma}^{(i)}(x) (\check{\varphi}^+(x) \partial_{\alpha} \check{\varphi}^-(x) - \partial_{\alpha} \check{\varphi}^+(x) \check{\varphi}^-(x)) \\ &\equiv \int_{\Sigma} d\sigma(x) q_{\Sigma}^{(i)}(x) \check{N}(x) \end{aligned} \tag{88}$$

of position of the quasiparticle on $\Sigma(t)$ observed by means of three spatial coordinate scalar functions $q_{\Sigma}^{(i)}(x)$, which satisfy the conditions

$$\partial^{\alpha} \Sigma \partial_{\alpha} q_{\Sigma}^{(i)} = 0, \quad \text{rank} \|\partial_{\alpha} q_{\Sigma}^{(i)}\| = n, \tag{89}$$

and thus define a point on a given Cauchy hypersurface $\Sigma = \{x \in V_{1,3} | \Sigma(x) = \text{const}\}$. In the globally static $V_{1,n}$, their restrictions on a completely geodesic hypersurface Σ are just functions $q^{(i)}(\xi)$ that have been introduced in Section 5. (For the Cartesian coordinates in $E_{1,3}$, such an operator was considered by Polubarinov, 1973.) It is easy to see that QFT-operators $\check{Q}^{(i)}\{\check{\varphi}; \Sigma\}$ are unique sesquilinear (in $\check{\varphi}^{\pm}$) Hermitean forms in \mathcal{F} , which can be constructed from $\check{\varphi}^{\pm}$, $\partial^{\alpha} \Sigma \partial_{\alpha} \check{\varphi}^{\pm}$, and do not contain derivatives of $q_{\Sigma}^{(i)}(x)$. The QFT-observables introduced above are evidently sufficient to describe quantum dynamics of a single quasiparticle if there is no processes of quasiparticle creation and annihilation as in the case of a globally static $V_{1,n}$, or if these processes can be neglected. Such dynamics is just quantum mechanics of a quasiparticle the space of states of which is a subspace

of \mathcal{F} consisting of the vectors

$$|\varphi\rangle \stackrel{def}{=} \{\varphi, \varphi\}_\Sigma^{-1/2} \int_{\{A\}} d\mu(A) \{\varphi(\cdot; A), \varphi(\cdot)\}_\Sigma \check{c}^+(A) |0\rangle, \quad (90)$$

determined by the field configuration

$$\Phi^- \ni \varphi(x) = \int_{\{A\}} f \mu(A) \{\varphi(\cdot; A), \varphi(\cdot)\}_\Sigma \varphi(x; A).$$

Obviously $\langle \varphi | \varphi \rangle = 1$. The QM-observables are determined by the matrix elements of the introduced QFT-operators between two one-quasiparticle states $|\varphi_1\rangle$ and $|\varphi_2\rangle$.

6.3. One-Particle States and Observables

There are infinitely many decompositions (82) and they generate Fock representations of the canonical commutation relations of the quantum field which are unitarily unequivalent in general. The main problem is to distinguish a subspace Φ^- in the space Φ_c of solutions of the field Eq. (81) for which the introduced formal quantum mechanics of a quasiparticle corresponds to the geodesic motion in $V_{1,n}$ and, therefore, may be called quantum mechanics of a *particle*. In the general $V_{1,n}$, this problem can be solved only as a quasinonrelativistic asymptotic approximation to the formal scheme, since the formally exact relativistic quantum mechanics can be constructed only in the globally static $V_{1,n}$ (see below). Therefore, we take as Φ^- a space Φ_L^- of the following asymptotic in c^{-2} solutions of Eq. (81)

$$\varphi_L(x) = \sqrt{\frac{\hbar}{2mc}} \exp\left(-i\frac{mc}{\hbar} S_\Sigma(x)\right) \hat{V}_L(x) \psi(x). \quad (91)$$

The notation here needs detailed explanations.

$S_\Sigma(x)$ is a solution of the Hamilton–Jacobi equation

$$\partial_\alpha S_\Sigma \partial^\alpha S_\Sigma = 1, \quad (92)$$

which satisfies the initial conditions $S_\Sigma(x)|_\Sigma = 0$ and $(r^\alpha(x) \partial_\alpha S_\Sigma(x))|_\Sigma > 0$ for any timelike vector field $\tau^\alpha(x)$ directed into the future. Any hypersurface $S_\Sigma(x) = \text{const}$, denoted further simply as S , is a level surface of the normal geodesic flow through Σ .

$\psi(x)$ is a solution of Schrödinger equation

$$i\hbar c \left(\partial^\alpha S \partial_\alpha + \frac{1}{2} \square S \right) \psi(x) = (\hat{H}_L^{(\text{ft})} + O(c^{-2(L+1)})) \psi(x), \quad (93)$$

$$\hat{H}_L^{(\text{ft})} \stackrel{def}{=} \hat{H}_0^{(\text{ft})} + \sum_{l=1}^L \frac{\hat{h}_n}{(2mc^2)^n}, \quad (94)$$

$$\hat{H}_0^{(ft)} \stackrel{def}{=} -\frac{\hbar^2}{2m} \left(\Delta_S(x) - \zeta R(x) + \left(\frac{1}{2}(\partial S \partial \square S) + \frac{1}{4}(\square S)^2 \right) \right), \quad (95)$$

and \hat{h}_l are differential operators which are determined by certain recurrence relations starting with $H_0^{(ft)}$ and contain only derivatives along the hypersurface S (“spatial derivatives”).

$\hat{V}_L(x)$ is an asymptotical differential QM-operator along S

$$\hat{V}_L(x) \equiv \hat{\mathbf{1}} + \sum_{l=1}^L \frac{\hat{V}_l}{(2mc^2)^l} + O(c^{-2(L+1)}), \quad (96)$$

where the operators \hat{v}_l are determined by the asymptotic relation

$$\{\varphi_1, \varphi_2\}_S = (\psi_1, \psi_2)_S \stackrel{def}{=} \int_S d\sigma \bar{\psi}_1 \psi_2 + O(c^{-2(L+1)}), \quad \varphi_1, \varphi_2 \in \Phi_L^-. \quad (97)$$

It provides Φ_L^- with the structure of $L^2(S; \mathbb{C}; \sigma)$ and ψ by the standard Born probabilistic interpretation in each configuration space S , i.e. $|\varphi(x)|^2$ is the probability density to observe the field configuration which may be called “a particle” at the point x belonging to the given hypersurface S . At least, this field configuration satisfies an intuitive idea of what is the quantum particle as a localizable object.

Thus, we have defined the space of states of a particle and can calculate the asymptotical one-to-one-particle transition probability amplitudes of form $\langle \varphi_1 | \check{O} | \varphi_2 \rangle$ for the QFT-operators of the observables defined earlier. To this end, each time when “the time derivative” $\nabla^\alpha S \nabla_\alpha$ appears in calculations, it should be substituted by the differential operator along S of the appropriate order determined by the Schrödinger Eq. (93). In effect, this completes the deduction of quantum mechanics of the particle in $V_{1,n}$ from quantum field theory in the quasi-nonrelativistic approximation because we have the matrix elements of observables of the particle, which were considered in Section 2. However, to compare the QFT-results with those of the canonical quantization, we need the operator representations of the observables as differential operators in $L^2(S; \mathbb{C}; \sigma)$. They are defined up to an asymptotically unitary transformation by the following general relation:

$$\langle \varphi_1 | \check{O} | \varphi_2 \rangle = (\psi_1, (\hat{O})_L \psi_2)_S \stackrel{def}{=} \int_S d\sigma \bar{\psi}_1 (\hat{O})_L \psi_2 + O(c^{-2(L+1)}), \quad \varphi_1, \varphi_2 \in \Phi_L^- \quad (98)$$

$$(\hat{O})_L \stackrel{def}{=} (\hat{O})_0 + \sum_{l=1}^L \frac{\hat{o}_l}{(2mc^2)^l}, \quad (99)$$

where \hat{O} is any of the QFT-operators and, again, \hat{o}_n are differential QM-operators along S determined by recurrence relations starting with $(\hat{O})_0$. From Eq. (98), it

follows that

$$(\hat{N})_L = \hat{\mathbf{1}} + O(c^{-2(L+1)}). \quad (100)$$

For other observables, from Eqs. (88), (87), and (99), one has the following non-relativistic QM-operators for further calculations of relativistic corrections: *the particle position on S*

$$(\hat{q}_S^{(i)}(x))_0 \stackrel{def}{=} q_S^{(i)}(x) \cdot \hat{\mathbf{1}}, \quad (101)$$

the particle momentum along $K_{(j)}^\alpha = \{0, K_{(j)}^i\}$ where $K_{(j)}^i$ is defined as in Eq. (76)

$$(\hat{p}_{(j)}(x))_0 \stackrel{def}{=} (P_{K_{(j)}})_0 = i\hbar \left(K_{(j)}^\alpha \nabla_\alpha + \frac{1}{2} \nabla_\alpha K_{(j)}^\alpha \right), \quad (102)$$

and the particle energy

$$(E(x))_0 \stackrel{def}{=} (P_K)_0|_{K^\alpha=c\partial^\alpha S} = \hat{H}_0^{(ft)}. \quad (103)$$

It is remarkable that not only nonrelativistic expressions for the energy QM-operator originated by the energy–momentum tensor $T_{\alpha\beta}$ and for the Hamiltonian in the Schrödinger equation (93) coincide but also their asymptotic representations of any order L are *asymptotically unitary equivalent* (see Tagirov, 1999).

6.4. Quantum Mechanics in the Globally Static Space-Time and Deformation of Canonical Commutation Relations

Pass now to the case of globally static $V_{1,n}$ and consider it, as in Sections 2–5, in a system of coordinates $\{x^\alpha\} \sim \{t, \xi\}$ in which $\omega_{ij}(t, \xi)\omega_{ij}(\xi)$. Then, the asymptotic expansions above can be represented in the formal closed forms

$$\hat{H}_\infty^{(ft)} = mc^2 \left(\left(1 + \frac{2\hat{H}_0^{(ft)}}{mc^2} \right)^{1/2} - 1 \right), \quad \hat{H}_0^{(ft)} = -\frac{\hbar^2}{2m}(\Delta_S - \zeta R), \quad (104)$$

$$\hat{V}_\infty = \left(1 + \frac{2\hat{H}_0^{(ft)}}{mc^2} \right)^{-1/4}, \quad (105)$$

$$\begin{aligned} (\hat{p}_{(j)})_\infty(x) = & -\frac{i\hbar}{2} \hat{V}_\infty^{-1} \cdot (K_{(j)}^i \nabla_i) \cdot \hat{V}_\infty + \frac{i\hbar}{2} \hat{V}_\infty \cdot (K_{(j)}^i \nabla_i)^\dagger \cdot \hat{V}_\infty^{-1} \\ & - \frac{\hbar\zeta}{2mc^2} \hat{V}_\infty \cdot (c\partial^\alpha S \nabla_\alpha)(\nabla_r K^r(j)) \cdot \hat{V}_\infty \end{aligned} \quad (106)$$

$$c(\hat{p}_{\partial S})_\infty(x) = mc^2 \left(1 + \frac{2\hat{H}_0^{(ft)}}{mc^2} \right)^{1/2} \text{ (energy operator)}, \quad (107)$$

$$(\hat{q}_S^{(i)})_\infty = q_S^{(i)}(x) + \frac{1}{2} [[\hat{V}_\infty, q_S^{(i)}(x)], \hat{V}_\infty^{-1}]. \quad (108)$$

Recall that we use ∇_α and ∇_i to denote the covariant derivative with respect to the metric tensors $g_{\alpha\beta}$ and ω_{ij} respectively.

6.5. QFT-Approach vs. Quantization of Geodesic Motion

What conclusions can be made from the formulae obtained just now comparing them with the results of Sections 2 and 3?

1. They are diffeoinvariant in V_n and $V_{1,n}$ owing to introduction of the functions $q_S^{(i)}(x)$, which were proposed in Section 5 to separate background coordinates ξ on V_n (that is on S) from the coordinates $q_S^{(i)}$ in terms of which a position on S of the quantum particle is observed.
2. The relativistic Hamiltonian $\hat{H}_\infty^{(ft)}$ is expressed through the nonrelativistic one $\hat{H}_0^{(ft)}$ just by formula (11) and supports the asymptotic meaning of quantization of $H(p, \xi)$, Eq. (7).
3. The nonrelativistic Hamiltonian $\hat{H}_0^{(ft)}$ is similar to DeWitt's one $\hat{H}_0^{(DW)}$, Eq. (36), but the coefficient before the scalar curvature R is an arbitrary constant ζ in $\hat{H}_0^{(ft)}$ instead of value $(1/6)$ in $\hat{H}_0^{(DW)}$. As it has already been said, the latter distinguished value of ζ corresponds to the conformal coupling of φ to gravitation, but only when $n = 3$. Another interesting difference is that $\hat{H}_0^{(ft)}$ is an exact expression with no other quantum potential terms if $c^{-1} = 0$ while $\hat{H}_0^{(DW)}(\xi)$ is the quasiclassical approximate expression. This difference is very interesting and poses a question: the nondiffeoinvariant part of quantum potential, is it a deficiency of the quantization of mechanics or is its absence in the QFT-approach a manifestation of some incompleteness of the canonical quantization of field?
4. The most remarkable consequence of the QFT-approach is that the position operators $(\hat{q}_S^{(i)})_N$ do not commute among themselves except the case of $N = 0$ and the same takes place for the momentum ones $(\hat{p}_{(j)})_N$. Therefore, the canonical commutation relations (15) are fulfilled only in the exactly nonrelativistic case $c^{-1} = 0$ and *the quasi-nonrelativistic commutation relations of primary observables are a deformation of the nonrelativistic ones*. An analogous deformation of the $\mathfrak{o}(3)$ algebra of the spin 1/2 operators arises when the QFT-approach is used for the Dirac particles (Tagirov, 1996).
5. The QFT-approach gives at once a quasi-nonrelativistic quantum mechanics in the general $V_{1,n}$ and arbitrary frame reference formed by the normal geodesic flow through an arbitrary Cauchy hypersurface S . In contrast, quantization of mechanics is formulated earlier only in the globally static and topologically elementary V_n and only in the frame of reference in which the metric tensor is time independent; this frame is formed by the

Killing flow. A consecutive formulation in the latter case needs a special study.

6. Moreover, since the theory is formulated actually in terms of matrix elements of the form (99), it may be applied to V_n of any topology.
7. The QFT-approach opens a way to formulate quantum mechanics of particles with nonzero spin, for which there are no classical counterparts.

6.6. Space Quantization in the Friedmann–Robertson–Walker Universe

At last, I like to announce an interesting result obtained in the QFT-approach for the case of the Friedmann–Robertson–Walker universe and a natural frame of reference in it, in which

$$ds^2 = c^2 dt^2 - b^2(t)\tilde{\omega}_{ij}(\xi) d\xi^i d\xi^j, \quad i, j, \dots = 1, 2, 3. \quad (109)$$

Let $q_{S \sim t_0}^{(i)}(\xi) = X^{(i)}(\xi)$ be the normal Riemannian coordinates which are measured in the units of the cosmological scale factor $b(t)$. In the standard Euclidean vector notation $\{X^{(1)}, X^{(2)}, X^{(3)}\} \equiv \vec{X}$ (see, for instance, Weinberg, 1972, chap. 13), one has

$$\tilde{\omega}_{ij} d\xi^i d\xi^j = (d\vec{X})^2 + \frac{k(\vec{X} \cdot d\vec{X})}{1 - k\vec{X}^2},$$

where $k = 1, 0, -1$ for the spatially spherical, flat, and hyperspherical universes, respectively.

Since the space geometry depends on the cosmological time t , the structure of quasi-nonrelativistic quantum mechanics and, thus, the notion of a particle are specified by an initial moment t_0 in which the Cauchy problem for the Schrödinger equation is posed. In the first nonvanishing order in C^{-2} , which is $O(C^{-4})$ for any $\hat{q}_{S \sim t_0}^{(i)}(\xi)$ in any $V_{1,n}$. For the normal Riemannian coordinates, one has in this order

$$[(\hat{X}^{(i)})_2, (\hat{Y}^{(j)})_2] = -k \left(\frac{\lambda_C}{b(t_0)} \right)^4 \left(X^{(i)} \frac{\partial}{\partial Y^{(j)}} - Y^{(i)} \frac{\partial}{\partial X^{(j)}} \right) + O(c^{-6}), \quad (110)$$

where $\lambda_C = \hbar/2mc$ is the Compton wavelength of the particle. It is remarkable that relation (110) is the $\mathfrak{o}(3)$ -part of the basic formula in Snyder's theory of quantized Minkowsky space-time (Snyder, 1947):

$$[\hat{x}^\alpha, \hat{y}^\beta] = l_0^2 L^{(\alpha\beta)}$$

where $L^{(\alpha\beta)}$ are the Lorentz group generators, x^α are the ordinary Cartesian coordinates and l_0 is an elementary length. According to relation (110), *the space seems to be quantized in principle in the standard theory* with no additional hypotheses, except the case of spatially flat universe, i.e., $k = 0$, and the elementary length $l(t_0) = (\lambda_C/b(t_0))\lambda_C$ depends on the moment of time in which the Schrödinger representation is specified. However, one should remember that a particle specified by

the moment t_0 has to be sufficiently heavy, the quasi-nonrelativistic approximation to be valid, and the processes of particle creation and annihilation caused by the time dependence of the cosmological factor $b(t)$ to be negligible. To conclude finally, may the space quantization have at least, a hypothetical physical sense, or, is it an artefact of the approximation, it is necessary also to estimate contributions of the next order in c^{-2} and, in the case of $k = 1$, to take into account singularity at $\bar{X}^2 = 1$. This author hopes to present such study in future elsewhere.

The most important point is, however, that the deformation of the canonical commutation relations and, consequently, the space quantization disappear for any L and t_0 in the exceptional case $k = 0$, i.e., in the spatially flat universe (see the general proof in Tagirov, 2000). This fact correlates remarkably with that, according to the modern astrophysical observational data, the real Universe is spatially flat with fantastically high accuracy which needs to be explained (the so-called problem of flatness). Could the last result not to be a quantum–mechanical reason for the flatness?

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